Analysis of Timescales of Response of a Simple Climate Model

ROBERT E. DICKINSON AND KIMBERLY J. SCHAUDT

Institute of Atmospheric Physics, The University of Arizona, Tucson, Arizona

(Manuscript received 16 September 1996, in final form 20 May 1997)

ABSTRACT

A detailed analysis is made of various analytic solutions to a 0D climate model coupled to an upwelling diffusion ocean model. In particular, the responses to impulse, step-function, exponential, and linear forcings are addressed. The coupled climate–ocean model is characterized by two timescales: a relatively fast timescale, τ_c, associated with the damping of an ocean temperature anomaly by thermal radiation down to the depth of the thermocline, and a slow timescale, τ_u, representing the time required for upwelling to balance vertical diffusion in an uncoupled ocean model. A small parameter, $\varepsilon = \tau_c/\tau_u$, is introduced, and it is shown from the analytic solutions that the decay of transients in the complex system is governed by a third timescale resulting from a combination of the first two timescales and written as $\varepsilon\tau_c$; that is, it is much faster than either of the more obvious timescales. The parameter ε evidently indicates the ratio of the depth of heating anomaly penetration to the depth of the thermocline. Examples involving forced solutions also indicate that this is the appropriate timescale for model response provided the forcing is faster than τ_u. For forcing timescales that are long compared to τ_u, τ_c becomes the timescale characterizing the lag. The response of the model on the timescale $\varepsilon\tau_c$ is equivalent to a near balance being reached between the radiative damping of the transient temperature anomaly and the downward diffusion of heat into the ocean. Inasmuch as vertical diffusion is an unphysical and questionable characterization of vertical energy exchange processes in the near-surface ocean, it is concluded that the actual ocean controls of climate response time are still poorly described.

1. Introduction

Climate models of the coupled land–ocean–atmosphere system are becoming increasingly complex and realistic. However, the most advanced such models are still incomplete with many documented shortcomings. Because of the nature of the scientific process and the complexities of the system being modeled, this may always be the case. More importantly, the most complex atmosphere–ocean–land models are not yet readily usable for integrated assessment studies of the end-to-end problem of climate change from increasing greenhouse gases. Such studies require long time integrations with a wide variety of additional models or assumptions about future growth of greenhouse gases, the national economic processes that are responsible for them, and possible curtailment strategies to reduce emissions.

There is a hierarchy of simpler models (Schneider and Dickinson 1974), including 1D (vertical column) radiative convective and 1D energy balance (latitudinal coordinate), that represents some of the features of the more complicated models adequately for some purposes. Since the 1970s, an even simpler 0D model has been used with the state of the atmosphere–surface climate characterized by a single temperature, T, that captures many of the properties of the earlier 1D models. This model has been used to describe feedbacks of 3D models (e.g., NAS 1979; Dickinson 1982) or to study the transient response of the ocean to perturbations (Hoffert et al. 1980; Dickinson 1981; Hansen et al. 1984; Harvey and Schneider 1985; Wigley and Schlesinger 1985; Watts and Morantine 1990; Kim and North 1991; Kim et al. 1992). More recently, Dutton (1995) has provided a thorough mathematical and physical analysis of many properties of this system. Simple climate models have had an important role in integrated assessment analyses, for example, as in the studies of Wigley and Raper (1987), who employ them for assessing the question of sea level rise, and in Houghton et al. (1990, 1995), which has made such models the quantitative framework for estimating the time evolution of global temperature in response to various scenarios of future greenhouse gas forcing.

Most past studies have used numerical approaches. Such integrations simply provide graphical information as to the behavior of the system. However, analytic solutions may provide more useful insights into the dependence of the solutions on underlying parameters (e.g., Dickinson 1981), hence their dominant physical processes.

An important question to be asked is, what determines

Corresponding author address: Dr. Robert E. Dickinson, Institute of Atmospheric Physics, The University of Arizona, P.O. Box 210081, Tucson, AZ 85721.
E-mail: robted@air.atmo.arizona.edu

the timescale of the response of the climate–ocean system to external perturbations, such as greenhouse warming? Timescale is traditionally expressed as the *e*-folding time for the decay of transient impulse or as the time required to adjust to a new steady-state solution. Since the forcing of greenhouse warming is gradually increasing, an alternative in this case has been to describe the time by which the transient solution lags the steady-state solution, or the fraction of the steady-state solution represented by the transient solution.

Simple models and many ocean GCMs represent vertical transports over the bulk of the ocean by vertical eddy diffusion and by largescale upwelling processes, essentially the viewpoint of Munk (1966). In this paper, we use a commonly assumed value for a vertical diffusion coefficient of $k = 1$ cm^2 s^{-1}. Current investigations indicate that the bulk of the transport in the interior ocean may be, in reality, by movement along constant density surfaces and that the coefficient for cross isopycnal diffusion may be an order of magnitude less (e.g., the measurement of Ledwell et al. 1993), except possibly in regions of boundary currents or seamounts.

The characteristic timescale of ocean response to vertical diffusion alone is simply D^2k^{-1}, where D is some characteristic length scale. For diffusion from the ocean surface, as the only transport process, D can refer to any given depth, since timescale depends on the depth of penetration. If upwelling is included with a vertical velocity w, then D can be equated to kw^{-1}.

The simplest 0D climate model that can be added to an ocean model, as used here, balances net radiation of the atmosphere–surface system with eddy heat diffusion into the ocean. Ocean surface temperatures change in time with this balance until energy transport into the ocean reaches a steady state. (Assumptions for this include that timescales are sufficiently long that the atmosphere is in steady state and that the atmosphere–surface system has equilibrated, so that radiative fluxes at the top of the atmosphere match net energy flux at the surface for a global average.) The temperature-dependent part of the radiative cooling is most simply characterized by a damping term linear in temperature, with its coefficient a sensitivity parameter in units of W m^{-2} K^{-1}. The coupling of ocean–surface temperature to radiative cooling suggests a second timescale that is the ratio of the specific heat of the depth of ocean to which steady-state heating penetrates (the ‘‘thermocline’’) divided by the climate damping parameter. Several authors have at various times noted this timescale and its importance for determining the climate response in the context of a single-slab ocean (e.g., Hansen et al. 1984; Dutton 1995). In the present paper, we consider the coupling of the upwelling diffusion ocean model to the 0D climate model. For this coupled climate model, the presence of these two timescales raises interesting questions: Which of these timescales is more important for determining the response, or are both important? Does the response depend simply on these two timescales or on some multiplicative product of them, for example, as this paper shows, the square of one divided by another? The analysis of this paper indeed finds a timescale involving a combination of the two most obvious timescales. This time is short compared to that in which the upwelling term becomes important and suggests that the climate response is controlled by vertical diffusion. Since vertical diffusion is an inaccurate and unphysical characterization of vertical energy exchange in the near-surface ocean, the upwelling diffusion model may be inappropriate for further studies of transient climate response.

Because of the weakness of the simple model, our conclusions do not necessarily translate in any quantitative fashion into insights for more complex and presumably more realistic models. However, even if not, they still provide useful guidance as to questions that must be addressed in looking at solutions of the more elaborate models. Furthermore, a qualitative point made from this analysis is highly likely to carry over into more detailed models. That is, the response behavior of the coupled system involves combinations of processes in the atmosphere and ocean and cannot be inferred by any separate analysis of individual atmospheric or ocean models.

2. Models and timescales

The simple transient 0D climate model is written [following in part the notation of Dutton (1995)],

$$F(t, T) + aT = Q(t), \tag{1}$$

where $F(t, T)$ is the time-dependent uptake of heat into the ocean, T is global average surface temperature, a is a sensitivity parameter, and Q is a perturbation heat input. Here, $F(t, T)$ has been represented in two forms.

a. Slab ocean

The first form is for the slab ocean (e.g., Dickinson 1981):

$$F(t, T) = C\,\partial T/\partial t, \tag{2}$$

where C is the effective heat capacity of the ocean proportional to the depth, D, of ocean in contact with the atmosphere on the timescales being considered [several elaborations of models have been proposed, all essentially representing the ocean with multiple boxes with coupling coefficients, e.g., a surface mixed layer (Dickinson 1981) or polar reservoirs (Harvey and Schneider 1985)]. Associated with (1) and (2) is the climate damping timescale, $\tau_c = C/a$. Transient solutions to (1) for slab ocean heat uptake approach a steady state with departure terms proportional to $\exp(-t/\tau_c)$. Our further analysis explores whether a similar conclusion can be obtained for the diffusion-upwelling model.

b. Diffusion-upwelling ocean

In its simplest form, this is a 1D ocean model, where (1) applies as a surface boundary condition that assumes a vertical coordinate z measured upward to the surface at $z = 0$,

$$F(t, T) = kc\,\partial T/\partial z \quad \text{at} \quad z = 0, \tag{3a}$$

where c is the heat capacity of ocean water per unit mass and, in the interior of the ocean,

$$\partial T/\partial t + w\,\partial T/\partial z - k\,\partial^2 T/\partial z^2 = 0, \tag{3b}$$

where we use the same symbol $T = T(z)$ for the vertically varying temperature within the ocean.

Equations (3a)–(3b) have the steady-state solution (i.e., $\partial T/\partial t = 0$)

$$T(z) = (T_{\rm surf} - T_{\rm bot})\exp(\gamma z) + T_{\rm bot}, \tag{4a}$$

where

$$\gamma = w/k, \tag{4b}$$

and where $T_{\rm bot}$ is a ‘‘bottom water’’ temperature and $T_{\rm surf}$ is the surface temperature, either calculated from (1) or prescribed from other considerations, since (1) is intended for use only with small perturbations. Equation (4) is classic in dynamic oceanography as the simplest model of the vertical structure of the oceans (perhaps approximating reality in the Tropics and subtropics). For $k = 1$ cm^2 s^{-1} and $w = 4$ m yr^{-1} [essentially the values originally suggested by Munk (1966)], the depth of the thermocline for this model is $\gamma^{-1} \simeq 790$ m. Another important scale used in our analysis is τ_u, the timescale for upwelling to balance diffusion:

$$\tau_u = k/w^2. \tag{5}$$

For the assumed values of $w = 4$ m yr^{-1} and $k = 1$ cm^2 s^{-1}, $\tau_u = 197$ yr. In other words, two centuries are required for vertical diffusion to balance upwelling and so to warm the ocean to the depth of the main thermocline (first km or so). This is also the timescale for upwelling to move water from the base of the thermocline to the surface.

The timescale τ_u is as long as or longer than the timescale for growth of greenhouse warming and certainly longer than the climate damping timescale, τ_c, which for the diffusion-upwelling ocean is given by

$$\tau_c = C/a = cD/a = ck/(wa). \tag{6}$$

For the thermodynamic properties of seawater, 7.8 m of water has unit heat capacity in W yr m^{-2} K^{-1} (Dickinson 1981). Multiplying the inverse of this value by a factor of 0.7 to allow for the global fractional area of oceans gives, for specific heat, the value $c = 0.090$ W yr m^{-3} K^{-1}. Then $C = cD = 71$ W yr m^{-2} K^{-1}. We take $a = \lambda$ W m^{-2} K^{-1}, where λ is a nondimensional sensitivity parameter, $\lambda = 1$ for a ‘‘sensitive climate’’ (4° warming for doubling of CO_2) or $\lambda = 2$ for an ‘‘insensitive climate’’ (2° warming for doubling of CO_2); this timescale is

$$\tau_c = \frac{71}{\lambda}\ \text{yr}. \tag{7}$$

3. Responses of the diffusion-upwelling model

A key factor in examining the time-dependent climate response is the ratio of the climate to upwelling timescales. Anticipating on the basis of the numerical values already suggested that this will normally be a small parameter, we denote it ε

$$\varepsilon = \tau_c/\tau_u = cw/a. \tag{8}$$

For the quoted values, its numerical value is $\varepsilon = 0.36$ for sensitive climate and 0.18 for insensitive climate values of λ.

a. Exponential forcing

This simple example has an additional timescale of the forcing but illustrates this paper's theme that the timescales for climate change tend to be fast because of dependence on the parameter ε. Assume exponential growth of $Q(t)$ in (1), that is, $Q = Q_0 \exp(\alpha t)$. An approximate fit to recent past history suggests a value of $\alpha = 0.01$ yr^{-1}. Such exponential growth is not very realistic for greenhouse gas heating as a whole but is assumed here to most simply illustrate the concept of time lag or fraction of the steady-state solution. It would be somewhat more realistic to assume that the greenhouse gases were increasing exponentially. The heating is sublinear in their concentrations; and, in particular, the effect of CO_2 is approximately proportional to the logarithm of its concentration, so that the assumption of an increase linear in t would be more realistic and is addressed in section 3d. Neglecting possible start-up transients, we assume solutions to (3b) proportional to $\exp(\alpha t)$ and find that the solution consists of terms proportional to $\exp(\gamma_+ z)$ and $\exp(\gamma_- z)$, where $\gamma_\pm$ is

$$\gamma_\pm = \frac{w}{k}\left[\frac{1}{2} \pm \left(\frac{1}{4} + \alpha\tau_u\right)^{1/2}\right]. \tag{9}$$

Only the γ_+ term is kept, so that the solution remains finite. At $z = 0$, the response of the diffusion-upwelling ocean to exponentially growing heating can be written

$$T = RT_{\rm eq}/(1 + \varepsilon), \tag{10}$$

where $T_{\rm eq} = Q/a$ is the exponentially growing response in the absence of heat uptake by the ocean. The response factor R in (10) is the ratio of the exponentially growing response to that for steady forcing, that is,

$$R = \frac{1 + \varepsilon}{1 + \varepsilon\left[\frac{1}{2} \pm \left(\frac{1}{4} + \alpha\tau_u\right)^{1/2}\right]}, \tag{11}$$

as obtained from substituting (3a) into (1) and substituting (4b) and (9) for the vertical derivatives for the steady-state solution, and (8) to define ε. For the previously quoted values, $R = 0.79$ (for sensitive) and 0.87 (for insensitive), the time lag, defined as the time at which $R \exp(\alpha t) = 1$, is $t_\ell = -(\ln R)/\alpha = 23$ yr (sensitive) and 14 yr (insensitive) (i.e., the number of years that would be required to reach the amplitude of the response obtained from applying an equilibrium assumption instantaneously).

For $\varepsilon \to 0$, two asymptotic limits can be inferred from (11), depending on whether $\alpha\tau_u \gg 1/4$ or $\alpha\tau_u \ll 1/4$; that is, $t_\ell \to \varepsilon(\tau_u/\alpha)^{1/2}$ or $t_\ell \to \varepsilon\tau_u$. For the selected example, with $\alpha\tau_u \simeq 2$, the former is closer but is somewhat of an overestimate because of the finite values of ε. Together, these suggest a time lag of the form $b(\tau_u)\tau_c$, noting that $\varepsilon\tau_u = \tau_c$, and where $b(\tau_u)$ is a term somewhat less than one, and approaching one for τ_u very large compared to both τ_c and α^{-1}. The $(1 + \varepsilon)^{-1}$ factor in (10) from the diffusion term represents the effect of upwelling of cold water acting to cancel in part the external heating Q. This oceanic energy sink is an artifact of the neglect, for simplicity, of atmospheric warming in high latitudes by oceanic bottom water formation.

The slab ocean defined by (2) gives a similar result, that is, in place of (11), a response factor of $R = 1/(1 + \alpha\tau_c)$, and so with the same time lag definition, there is a somewhat different dependence of the time lag on the growth parameter α and the climate timescale τ_c. The velocity w in the upwelling diffusion ocean is replaced by a velocity scale αD, where D is the depth of the slab ocean. To match the values for R estimated following (11), $D = 260$–300 m, or about 1/3 its value for the diffusion-upwelling ocean. However, the diffusion-upwelling and slab-ocean time lags match for the same value of D in the limit of $\alpha\tau_c$ and ε becoming very small.

b. Impulse forcing

The response to a point impulse provides important information on response timescales for decay from any disturbance from equilibrium and gives a Green's function for synthesis to any time-dependent forcing. Equations (1), (3a), and (3b) are nondimensionalized by substituting $t = \tau_u t^*$ and $z = kz^*\,w^{-1}$. Written in nondimensional starred variables they become

$$\varepsilon\frac{\partial T'}{\partial z^*} + T' = Q(t^*)/a \quad \text{and} \tag{12}$$

$$\frac{\partial T'}{\partial t^*} + \frac{\partial T'}{\partial z^*} - \frac{\partial^2 T'}{\partial z^{*2}} = 0, \tag{13}$$

where T' is the departure from the steady-state solution given by (4). We assume $Q(t^*) = a\delta(t^*)$, where $\delta(t^*)$ is a unit impulse delta function. A Laplace transform is performed (using s as the Laplace transform variable), and (ˆ) is used to denote Laplace-transformed variables. The transformed solution to (12) and (13) is dimensionless and written

$$\hat{T} = \frac{\exp(\gamma^* z^*)}{(\varepsilon\gamma^* + 1)}, \tag{14}$$

where γ^* is the nondimensional equivalent to the + term in (9),

$$\gamma^* = \left[\frac{1}{2} + \left(\frac{1}{4} + s\right)^{1/2}\right]. \tag{15}$$

The positive branch has been selected because of the usual requirement that solutions be bounded for the chosen contour of s (see the appendix).

Equation (14) is transformed back to the time domain by contour integration (appendix) into the solution

$$\begin{aligned} T' = {} & \varepsilon^{-1}\exp(0.5z^* - 0.25t^*) \\ & \times \left[\frac{1}{\sqrt{\pi t^*}}\exp(-z^{*2}/4t^*) + be^{-bz^*}e^{b^2t^*}\right. \\ & \left.\times \{\operatorname{erfc}(0.5z^*/\sqrt{t^*} - b\sqrt{t^*}) - 2\}\right], \end{aligned} \tag{16}$$

where $b = (1/2 + 1/\varepsilon)$. The -2 at the end of the last term in (16) is an artifact of the argument of the erfc being negative. It can be eliminated by using $+b$ and -0.5 in (16). Near the surface ($z \approx 0$), in the limit as $t \to 0$, this solution has the approximate behavior

$$\underset{t^*\to 0}{T'} \simeq \frac{\exp(-z^{*2}/4t^*)}{\varepsilon\sqrt{\pi t^*}}, \tag{17}$$

which is simply the initial diffusion wave. This initial decay is lost if smoother forcing is imposed. For large time and $z = 0$, using

$$\operatorname{erfc}(x) \simeq x^{-1}\pi^{-1/2}\exp(-x^2) \times \left[1 + \sum_{m=1}^{\infty}(-1)^m\frac{1\cdot 3\cdots(2m-1)}{(2x^2)^m}\right],$$

we have

$$\underset{t^*\to\infty}{T'} \simeq \frac{\exp\left(-\frac{1}{4}t^*\right)}{2\varepsilon b^2\sqrt{\pi t^{*3}}} \to 0. \tag{18}$$

This large time residual is the product of a slow decay term, $\exp(-0.25t^*) = \exp(-t/4\tau_u)$ on the diffusion timescale and an algebraic decay factor, which scales as $1/(\varepsilon b^2 t^{*3/2}) \simeq \tau_c\tau_u^{1/2}/t^{3/2}$. Note that the individual terms in (16) decay for large time as $t^{-1/2}$, but that these lowest order contributions in $t^{-1/2}$ cancel. More generally, for intermediate time, the second term's exponential in (16) is a function of $(b^2 - 1/4)t^* \approx t(\tau_u + \tau_c)/\tau_c^2$. Thus, the appropriate coupled model adjustment time

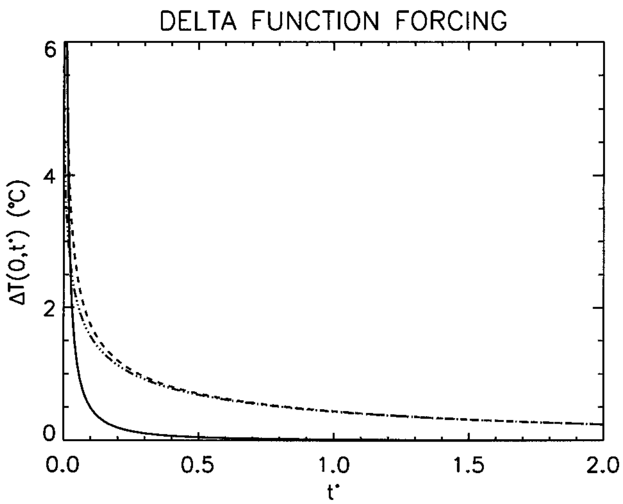

FIG. 1. The dashed line is the first term on the right-hand side of (16). The dash–triple-dot line is the negative of the second term. The solid line is the sum of the terms in (16), showing that the T response is much faster than that of its individual components.

is not τ_u or τ_c but a scale even faster than τ_c denoted τ_{0a}, where

$$\tau_{0a} = \frac{\tau_c^2}{(\tau_u + \tau_c)} \approx \begin{cases} 19 \text{ yr for sensitive} \\ 5 \text{ yr for insensitive,} \end{cases} \tag{19}$$

for the previously assumed numbers. Thus, in this and perhaps all transient adjustment solutions, τ_{0a} is the adjustment time. For $\tau_u \gg \tau_c$, (9) gives $\varepsilon \ll 1$, and $\tau_{0a} = \varepsilon\tau_c[1 + 0(\varepsilon)]$. The timescale $\varepsilon\tau_c$ represents the timescale of damping of the ocean thermal anomaly to a depth $\varepsilon D = kc/(\alpha w^2)$. Evidently this is the depth to which a transient surface heating anomaly can penetrate before it is damped by radiation.

Further graphical examination (see Fig. 1) of the two terms in (16) at $z = 0$ shows them to rapidly approach each other while t^* is still small; hence, (16) goes from approximate $t^{*-1/2}$ to $t^{*-3/2}$ behavior as t^* increases but is still small. The details of this algebraic behavior are sensitive to the details of the assumed source, whereas the rapid decay on timescale τ_{0a} from the small time to large time behavior should be independent of the detailed shape in time of the source.

c. Step-function forcing

This is illustrated by another perhaps more realistic question, the transient adjustment to steady-state $T_{\mathrm{eq}}/(1 + \varepsilon)$ given a suddenly (instantaneously) imposed heating. In place of (14), the Laplace transform solution is

$$\hat{T} = \frac{T_{\mathrm{eq}}\varepsilon^{\gamma^* z^*}}{s(\varepsilon\gamma^* + 1)}, \tag{20}$$

where γ^* is that given by (15). When its inverse is evaluated at $z \approx 0$, it becomes

$$T'(\varepsilon) = \frac{T_{\mathrm{eq}}}{1 + \varepsilon}\left[1 - \varepsilon g_1(t^*) + \frac{\varepsilon}{2}g_2(t^*)\right], \tag{21a}$$

where

$$g_1(t^*) = b \exp\left[\left(b^2 - \frac{1}{4}\right)t^*\right] \operatorname{erfc}(b\sqrt{t^*}) \tag{21b}$$

and

$$g_2(t^*) = \operatorname{erfc}(0.5\sqrt{t^*}). \tag{21c}$$

As $t \to \infty$, both g_1 and g_2 decay to zero, so the term in brackets in (21a) approaches 1; that is, (21a) approaches the steady-state solution. Approximating the error functions with their asymptotic behavior, we have

$$\underset{t^*\to\infty}{\varepsilon g_1} \simeq \frac{\varepsilon e^{-t^*/4}}{\sqrt{\pi t^*}}\left(1 - \frac{1}{2b^2 t^*}\right) \quad \text{and} \tag{22a}$$

$$\underset{t^*\to\infty}{\frac{\varepsilon}{2}g_2} \simeq \frac{\varepsilon e^{-t^*/4}}{\sqrt{\pi t^*}}\left(1 - \frac{2}{t^*}\right) \tag{22b}$$

showing that both (22a) and (22b) decay on a $4\tau_u$ timescale.

In the limit of small time,

$$g_1 \simeq b\left(1 - \frac{2b\sqrt{t^*}}{\sqrt{\pi}}\right) \quad \text{and}$$

$$g_2 \simeq \left(1 - \frac{\sqrt{t^*}}{\sqrt{\pi}}\right) \tag{23a}$$

so that as $t^* \to 0$, (21a) reduces to

$$T'(t^*) = \frac{2T_{\mathrm{eq}}}{\varepsilon}\sqrt{\frac{t^*}{\pi}}, \tag{23b}$$

and we see that the initial growth in response to forcing is on the fast timescale of $\varepsilon^2\tau_u = \tau_c^2/\tau_u \simeq \tau_{0a}$. Figure 2 shows the solution given in (21a).

d. Linear forcing

Perhaps the most realistic form of heating, as discussed earlier, is linear heating. The heating is in the form of

$$Q(t) = aT_0(\beta\tau_u t^* + 1), \tag{24}$$

where T_0 is a temperature scale and β is a linear heating rate in units of per year. Under these conditions, the Laplace transform that is to be solved is

$$\hat{T} = \frac{T_0\tau_u\beta e^{\gamma^* z^*}}{s^2(\varepsilon\gamma^* + 1)}, \tag{25}$$

where γ^* is defined in (15).

The solution to (25), evaluated at the surface ($z^* = 0$) is

$$T'(0, t^*) = \frac{\tau_u \beta T_0}{(\varepsilon + 1)}\left\{t^* - \frac{\varepsilon}{(\varepsilon + 1)} + \frac{\varepsilon}{2}\left[t^* + \frac{1}{\left(b^2 - \frac{1}{4}\right)} + 2\right] \operatorname{erfc}\left(\frac{\sqrt{t^*}}{2}\right) - \frac{b\varepsilon e^{(b^2 - 1/4)t^*}}{\left(b^2 - \frac{1}{4}\right)} \operatorname{erfc}(b\sqrt{t^*}) - \sqrt{\frac{t^*}{\pi}} e^{-t^*/4}\right\}. \quad (26)$$

Figure 3 illustrates (26) for a heating rate, β, of 0.036 yr^{-1} and $T_0 = 1\ \mathrm{W\ m^{-2}}$. The equation $\beta = 0.036$ is chosen to fit the exponential $\alpha = 0.01$ heating after τ_u years. For the same amount of heating (i.e., at the same time), the temperature change is larger for a sensitive climate than for the insensitive climate case. This is generally expected. The dashed lines in Fig. 3 show for comparison the temperature change obtained assuming instantaneous equilibrium. The horizontal difference between solid and dashed lines shows the time lag of the climate response, greater for the sensitive case than the insensitive case. Over the intermediate times, as illustrated in Fig. 3, the lags are remarkably short, again because of the ε control. At large enough times, the temperature approaches with a lag that of the steady-state solution with diffusion, that is, $a^{-1}(1 + \varepsilon)^{-1}Q$, and is

$$T'(0, t^*) \simeq \frac{\tau_u \beta T_0}{(\varepsilon + 1)}\left(t^* - \frac{\varepsilon}{(\varepsilon + 1)}\right) \quad \text{for } t^* \gg 1. \quad (27)$$

Figure 3 shows that, for smaller times, the time lag is less than this asymptotic value; in dimensional units, the limiting time lag is approximately $\varepsilon\tau_u = \tau_c$. That is, the most extreme time lag possible is that representing the time required for climate radiation to damp a thermal anomaly to the depth of the thermocline.

4. Discussion and conclusions

This study presents analytic solutions and their asymptotic analyses for the simplest ocean–atmosphere coupled climate model. The ocean is represented by a transient upwelling diffusion model, and the atmosphere by a 0D energy balance coupled to vertical heat diffusion at the ocean surface. The ocean model appears to be characterized by two timescales. The upwelling-diffusion timescale, τ_u, typically about 200 yr, gives the time required by an ocean model to equilibrate upwelling with vertical temperature diffusion. The climate timescale, τ_c, gives the time required for atmospheric radiative feedbacks to adjust ocean temperatures to the depth of the main thermocline. It is likely in the range of 30–80 yr, depending on climate sensitivity. However, a third timescale, $\tau_{0a} \simeq \tau_c^2/\tau_u$, has been identified as most important for characterizing the adjustment to disturbances in climate heating. This timescale has been written as $\tau_{0a} = \varepsilon\tau_c[1 + O(\varepsilon)]$, where $\varepsilon = \tau_c/\tau_u$ is regarded as a small parameter. It is important for the description of the adjustment to disturbances in climate heating on timescales short compared to τ_u and is likely in the range 0.15–0.4. It is interpreted as the ratio of the depth to which a heating transient can penetrate before being radiatively damped to the depth k/w of the thermocline. In other words, the fast timescale is achieved by involving a relatively shallow layer of ocean. Analytic solutions have been obtained for exponential, impulse, linear, and switch-on heating, and their small and large time asymptotic behavior examined.

Figure 4 compares times for particular fractions of steady state to have been achieved versus ε for step-function heating. For comparison, a slab ocean would approach a value of $1 - e^{-1} = 0.63$ of equilibrium for a time of ε, and $1 - e^{-1/2} = 0.4$ of equilibrium for a time of 0.5ε. Figure 4 shows that up to 50% of steady state is achieved for times much smaller than given by such slab estimates and that only for equilibration of 85% or greater does the required time become about as long as indicated by a slab estimate.

With ε as a small parameter, the solution to the step-function forcing indicates that for intermediate times, the solution does not adjust with either the τ_c or τ_u timescales but, on a shorter timescale, $\varepsilon\tau_c = \varepsilon^2\tau_u$. Graphical examination of the response to switch-on forcing indicates that this quadratic dependence on ε is found only for $\varepsilon \lesssim 0.2$ and a temperature change up to approximately 65% of total (long time) change, but that for large values of ε within its expected range response times for 50% of steady state remain small compared to τ_c; only when the equilibrium reaches at least 85% does the required time become comparable to $\tau_c = \varepsilon\tau_u$ (i.e., $t^* = \varepsilon$), that is, comparable to the response for a slab ocean.

For impulse forcing, the response time is even more rapid. Figure 5 illustrates the time, t^*_{10}, it takes for the total temperature change to be 10% of the first (larger) term in (16) as a function of ε. It also illustrates that, for all ε, the time t^*_{10}, is smaller than ε, and therefore it takes a shorter time than does the slab ocean to reach 63% of its final change.

For an example of forcing increasing in time, the lags of the solutions from their equilibrium responses were examined. For forcing timescales slow compared to τ_u, these lags correspond to that inferred from a slab ocean with depth of the main thermocline, but for relatively rapid changes in forcing, again the lag is equivalent to that of a much shallower layer being coupled to radiative damping.

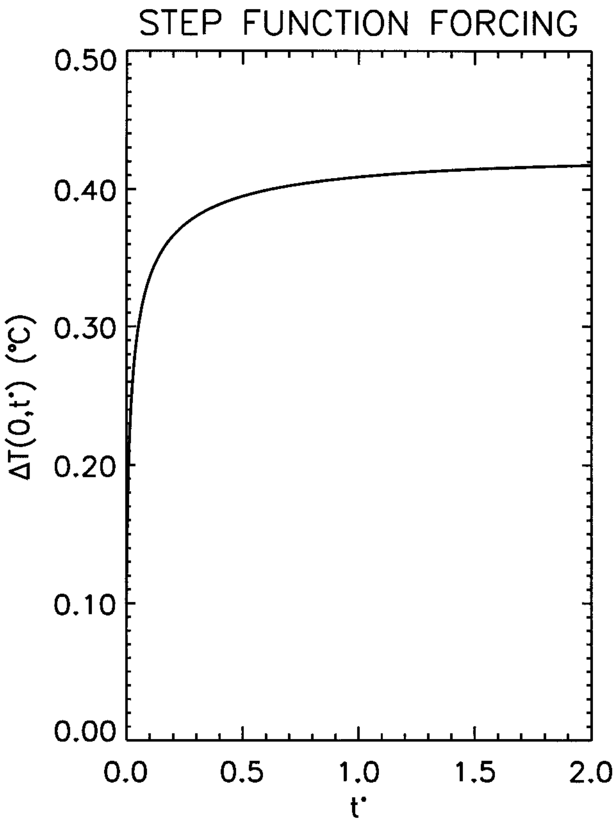

FIG. 2. The change in ocean surface temperature vs $t^* = t/t_u$ for a step heating (instantaneously imposed) with $Q_0 = 1$ and an insensitive climate. The temperature at large time approaches 0.423°C. This requires $t^* \gtrsim 2.5$ (i.e., *about 500 yr*).

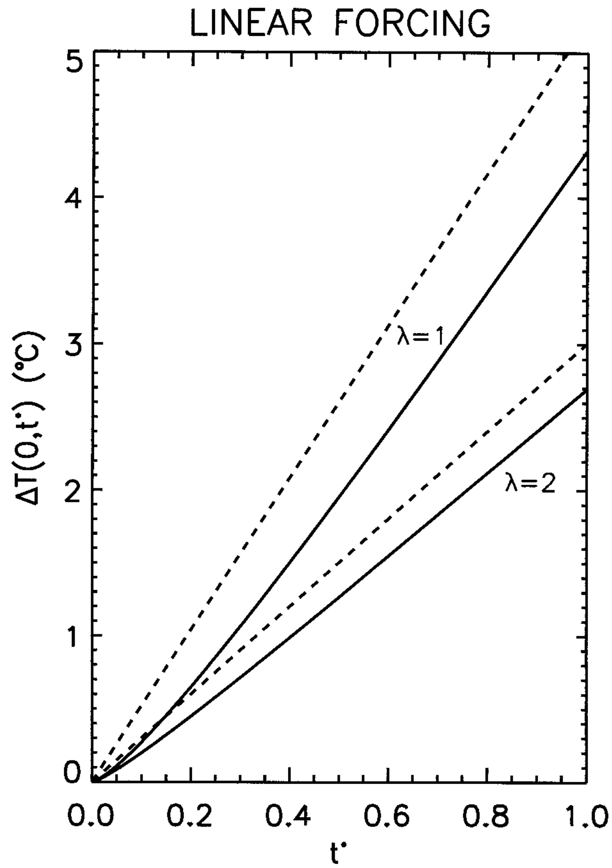

FIG. 3. The change in ocean surface temperature vs $t^* = t/t_u$, for linear heating. The upper set of solid and dashed lines is for the temperature and forcing (in units of temperature), respectively, for a sensitive climate, $\lambda = 1$. The lower set of solid and dashed lines represents temperature and forcing, respectively, for an insensitive climate, $\lambda = 2$. The heating rate, β, of (24) is 0.036 yr^{-1}, and $Q_0 = 1$ W m^{-2}.

The fast adjustment time τ_{0a} from (19) $\simeq \tau_c^2/\tau_u = c^2k/a^2$ is independent of the upwelling velocity w and is quadratic in the climate damping parameter a, a dependence previously noted from numerical integrations of the model used here by Hansen et al. (1985). The interpretation in terms of the model is disappointingly simple. That is, it is the response time implied by including only the diffusion and not the upwelling term in (3b). In this approximation, the thermohaline circulation becomes irrelevant to the timescale for temperature adjustment. This disappearance of the upwelling may be fortunate, in view of the much greater complexity of the real thermohaline circulation than assumed here, for example, as reviewed by Schmitz (1995) and Fine (1995), with large ''conveyor belt'' flows between oceans and upwelling apparently concentrated in a few places on the flanks of strong deep currents and with multiequilibria possible in coupled models (e.g., Saravanan and McWilliams 1995).

The assumed upwelling velocity of 4 m yr^{-1} applied to the total ocean surface is about 40 Svedrups, that is about double currently accepted values for the thermohaline circulation. Our analysis has no connection to the recognized importance of the thermohaline circulation in determining equilibrium climate, but it suggests that more detailed attention should be paid to vertical heat transfer in the near-surface layers of the ocean. An average mixed layer depth is simple to include and from our analysis clearly important in determining the lower limit of the response timescale. If large values of diffusion coefficients are to be found in regions of stable stratification, these are not yet known (Kunze and Sanford 1996; Caldwell and Moum 1995) but would be expected to occur, if anywhere, along boundaries or other topographic obstacles and/or where there are strong currents.

The general inference of this paper, that heat exchanges in the first several hundred meters of ocean depth are most important for determining the climate response time, suggests that this response may still be poorly quantified, perhaps even in the most advanced 3D models. In particular, what are the roles of strong, wind-driven coastal circulations; the winter deepening of the extratropical mixed layers to several hundred meters or more by cooling to the atmosphere; and the even deeper depression of the thermocline by Ekman pump-

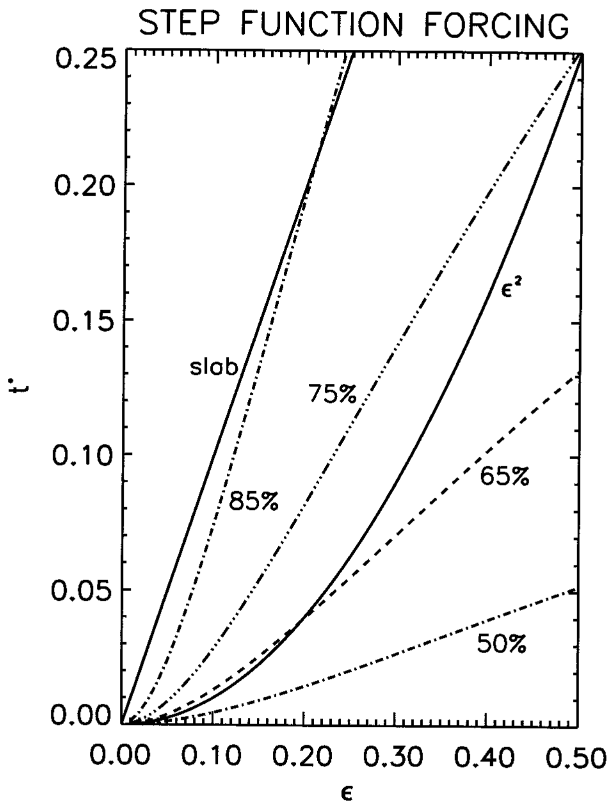

FIG. 4. Response time vs ε for various criteria. The solid curve on the right side is $t^* = \varepsilon^2$. The dash–dot line on the right side illustrates 50% of total temperature change. The dashed line illustrates 65% of total. The dash–triple-dot line illustrates 75% of total change. The dash–dot line on the left side illustrates 85% of total. The solid line on the left side illustrates the slab-ocean response.

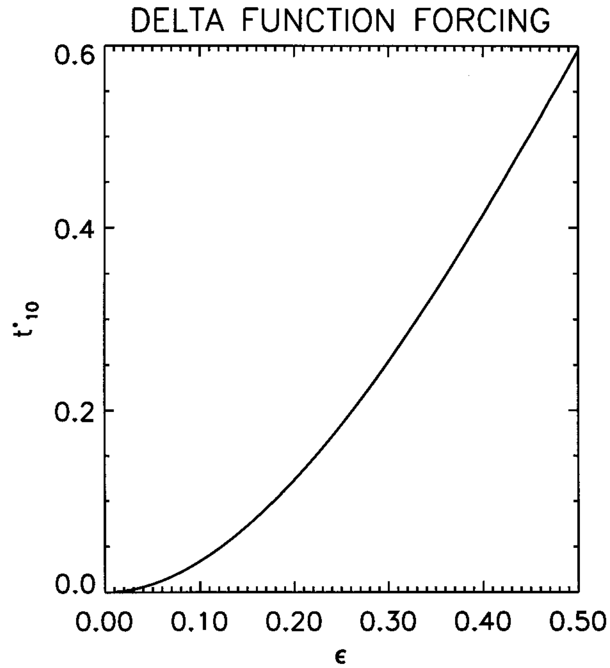

FIG. 5. Response time vs ε. The response time t_{10}^* is the time it takes for the total temperature (16) to be 10% of the first term in (16).

ing in the subtropical gyres? The Ekman pumping is accompanied by subduction of mixed layer water into the main thermocline (e.g., as discussed by Qiu and Huang 1995 and New et al. 1995), which in the subtropical North Atlantic and Pacific circulates as much or more water than the deep thermohaline circulation. In addition, the Southern Ocean has a very large wind-driven vertical circulation, indicated by numerical simulation (Danabasoglu and McWilliams 1995) to be several tens of svedrups in magnitude. All the above-mentioned processes, by acting to couple surface waters to deeper ocean layers, would act to slow the climate response from that indicated by the present analysis.

Thus, considerable uncertainty remains from the ocean viewpoint as to the initial adjustment of the complex climate system to forcing. Although it would appear that this might be much faster than expected from arguments derived from a slab model or uncoupled ocean, especially in the limit of an insensitive climate and over the first 70% or so of the adjustment, its quantification with the upwelling-diffusion model is weak. The dominance of the vertical diffusion process in determining the response timescale and the poor resemblance of actual exchange processes to such vertical diffusion suggests the inappropriateness of the upwelling diffusion model for transient climate studies.

Acknowledgments. We thank Qingqiu Shao, Zong-Liang Yang, and anonymous reviewers for helpful comments on the manuscript, and Margaret Sanderson Rae for editorial assistance and word processing. Funding for this study was provided by the National Science Foundation under Grant ATM-9419715 and NASA-UPN5429-82-22 and 429-81-22.

APPENDIX

Solution Details for Impulse Forcing

The differential equation (3b) is

$$\frac{\partial T}{\partial t} + w\frac{\partial T}{\partial z} - k\frac{\partial^2 T}{\partial z^2} = 0, \tag{A1}$$

and the boundary conditions are

$$ck\left.\frac{\partial T}{\partial z}\right|_{z=0} + aT|_{z=0} = Q(t), \tag{A2}$$

and $T(-\infty, t)$ remains finite. Using the starred parameters $t = \tau_u t^*$ and $z = (k/w)z^*$ with $\tau_u = k/w^2$ and with $\varepsilon = (wc/a)$, one obtains, for the differential equation (A1),

$$\frac{\partial T}{\partial t^*} + \frac{\partial T}{\partial z^*} - \frac{\partial^2 T}{\partial z^{*2}} = 0, \tag{A3}$$

and the boundary condition (A2) becomes

$$\varepsilon \frac{\partial T}{\partial z^*} + T = \frac{Q(\tau_u t^*)}{a} \tag{A4}$$

for an impulse heating function of

$$Q(\tau_u t^*) = a\delta(t^*) + Q_0, \tag{A5}$$

where Q_0 is the initial, steady-state heating. The Laplace transform of the differential equation (A3), with $\hat{T}$ denoting the transform, is

$$\frac{\partial^2 \hat{T}}{\partial z^{*2}} - \frac{\partial \hat{T}}{\partial z^*} - s\hat{T} = -T(z^*, 0). \tag{A6}$$

The right-hand side of (A6), once transformed back to real time, will produce the steady-state solution as shown in (4). The Q_0 is defined by the boundary condition (A4) and the steady-state solution, (4). The Q_0 in (A5) will produce the right-hand side of (A6); therefore, to simplify the solution, the pieces that are associated with the steady-state solution will be ignored. In effect, the solution found is only for the change in temperature.

The general solution to (A6) with the right-hand side removed is

$$\hat{T} = c_1 e^{\gamma_+ z^*} + c_2 e^{\gamma_- z^*}, \tag{A7}$$

where

$$\gamma_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} + s}. \tag{A8}$$

Only the γ_+ root is chosen to keep $\hat{T}$ finite as z goes to $-\infty$. The variable γ^*, in the main text, is γ_+ here in the appendix. The transform of the boundary condition (A4) with the Q_0 removed is

$$\varepsilon \left.\frac{\partial \hat{T}}{\partial z^*}\right|_{z^*=0} + \hat{T}|_{z^*=0} = 1. \tag{A9}$$

When the solution [(A7) with $c_2 = 0$] is placed into the boundary condition (A9), one obtains

$$\hat{T} = \frac{e^{\gamma_+ z^*}}{(\varepsilon\gamma_+ + 1)}. \tag{A10}$$

Performing the inverse transform to (A10) gives

$$\Delta T(Z^*, t^*) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{\gamma_+ z^*} - e^{st^*}}{(\varepsilon\gamma_+ + 1)}\, ds, \tag{A11}$$

where $\Delta T(z^*, t^*)$ is the change in temperature from the steady-state solution. The integral in the solution (A11), once some algebra is performed, is done by contour integration. The integral rewritten in explicit terms of s is

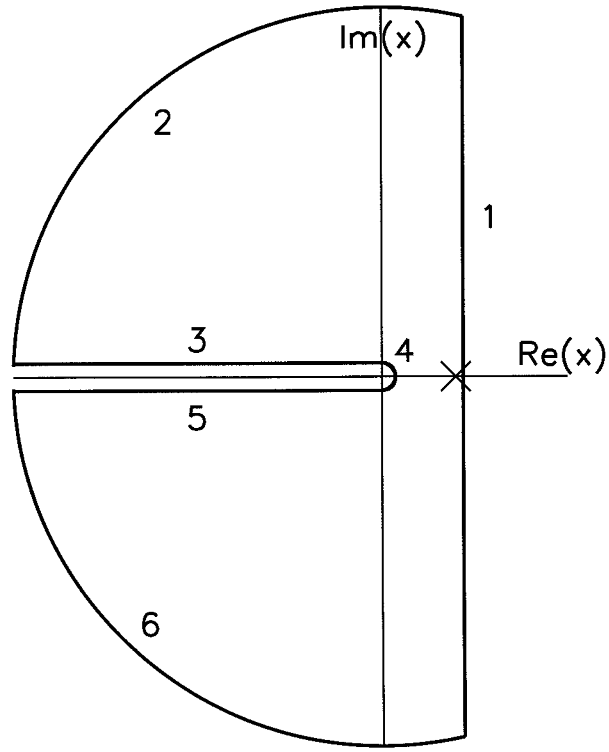

FIG. A1. The contour for integration of (A13). The direction of line 1 upward from $c' - i\infty$ to $c' + i\infty$. The direction of line 2 and line 6 is counterclockwise, and the direction of 4 is clockwise. The direction of line 3 is from left to right ($-\infty$ to 0). The direction of line 5 is from right to left (0 to $-\infty$). The large X on the positive, real x axis is the location of pole at $x = b^2$.

$$\begin{aligned} A &= \int_{c-i\infty}^{c+i\infty} \frac{e^{\gamma_+ z^*} - e^{st^*}}{(\varepsilon\gamma_+ + 1)}\, ds \\ &= \frac{e^{z^*/2}}{\varepsilon} \int_{c-i\infty}^{c+i\infty} \frac{e^{\sqrt{1/4+s}\,z^*} e^{st^*}}{\left(b + \sqrt{\frac{1}{4} + s}\right)}\, ds, \end{aligned} \tag{A12}$$

where $b = 1/2 + 1/\varepsilon$. Performing a change of variable of $x = 1/4 + s$ and completing the square in the denominator, A becomes

$$A = \frac{e^{z^*/2} e^{-t^*/4}}{\varepsilon} \int_{c'-i\infty}^{c'+i\infty} \frac{(b - \sqrt{x}) e^{z^*\sqrt{x}} e^{t^* x}}{(b^2 - x)}\, dx, \tag{A13}$$

where $c' \gtrsim b^2$. The solution to the integral (A13) is done by contour integration. [This contour is shown in Fig. A1. The integral is represented by line 1 in Fig. A1 and is equal to the residue minus the contributions from lines 2–6.] The residue of A is zero. Contributions from lines 2 and 6 are zero. The contribution from line 4 is also zero, which can be shown by the following:

$$\begin{aligned} \underset{\lim r\to 0}{\text{line 4}} &= \int_{\pi}^{-\pi} \frac{(b - \sqrt{r}\, e^{i\theta/2}) e^{z^*\sqrt{r}\, e^{i\theta/2}} e^{t^* r e^{i\theta}}}{(b^2 - re^{i\theta})} (ri)\, d\theta \\ &= 0, \end{aligned} \tag{A14}$$

where the substitution $x = re^{i\theta}$ is used, which is clearly zero when the limit $r = 0$ is taken. Line 3 uses the substitution $x = re^{i\pi}$, and line 5 uses the substitution $x = re^{-i\pi}$. When the exponent is expressed as -1, i or $-i$, as appropriate, the integrals become

$$\text{line 3} = \int_{\infty}^{0} \frac{(b - i\sqrt{r})e^{iz^*\sqrt{r}}e^{-rt^*}}{(b^2 + r)}(-dr) \quad \text{and} \tag{A15}$$

$$\text{line 5} = \int_{0}^{\infty} \frac{(b + i\sqrt{r})e^{-iz^*\sqrt{r}}e^{-rt^*}}{(b^2 + r)}(-dr). \tag{A16}$$

Using 1 = residue − 2 − 3 − 4 − 5 − 6, one obtains

$$\text{line 1} = 2i \int_{0}^{\infty} \frac{\sqrt{r}\,\cos(z^*\sqrt{r})e^{-rt^*}}{(b^2 + r)}\,dr - 2ib \int_{0}^{\infty} \frac{\sin(z^*\sqrt{r})e^{-rt^*}}{(b^2 + r)}\,dr. \tag{A17}$$

The denominator $(b^2 + r)$ is removed by the following method:

$$\begin{aligned}
\int_{0}^{\infty} \frac{\sin(z^*\sqrt{r})e^{-rt^*}}{(b^2 + r)}\,dr
&= e^{b^2t^*} \int_{0}^{\infty} \frac{\sin(z^*\sqrt{r})e^{-(r+b^2)t^*}}{(b^2 + r)}\,dr \\
&= e^{b^2t^*} \int_{\infty}^{t^*} \frac{\partial}{\partial t^*} \int_{0}^{\infty} \frac{\sin(z^*\sqrt{r})e^{-(r+b^2)t'}}{(b^2 + r)}\,dr\,dt' \\
&= -e^{b^2t^*} \int_{\infty}^{t} e^{-b^2t'} \int_{0}^{\infty} \sin(z^*\sqrt{r})e^{-rt'}\,dr\,dt'.
\end{aligned} \tag{A18}$$

The lower bound of ∞ for the time integral is chosen in order to have the integral at this lower bound equal to zero. Although this is not absolutely necessary, it makes the integral considerably easier to solve. The double integral is easy to perform using standard integration tables. The first term in (A17) reduces to

$$\int_{0}^{\infty} \frac{\sqrt{r}\,\cos(z^*\sqrt{r})e^{-rt^*}}{(b^2 + r)}\,dr = -e^{b^2t^*} \int_{\infty}^{t^*} - e^{b^2t'} \int_{0}^{\infty} \sqrt{r}\,\cos(z^*\sqrt{r})e^{-rt'}\,dr\,dt'. \tag{A19}$$

Equation (A19) is also easy to perform using the standard integration tables. The temperature change (A11) is then

$$\Delta T(z^*, t^*) = \frac{e^{z^*/2}e^{-t^*/4}}{\varepsilon}\left\{\frac{e^{-z^{*2}/4t^*}}{\sqrt{\pi t^*}} + be^{b^2t^*}e^{-bz^*} \times \left[\operatorname{erfc}\left(\frac{z^*}{2\sqrt{t^*}} - b\sqrt{t^*}\right) - 2\right]\right\}.$$

REFERENCES

Caldwell, D. R., and J. N. Moum, 1995: Turbulence and mixing in the ocean. *Rev. Geophys.,* **33** (Suppl.), 1385–1394.

Danabasoglu, G., and J. C. McWilliams, 1995: Sensitivity of the global ocean circulation to parameterizations of mesoscale trace transports. *J. Climate,* **8,** 2967–2987.

Dickinson, R. E., 1981: Convergence rate and stability of ocean–atmosphere coupling schemes with a zero-dimensional climate model. *J. Atmos. Sci.,* **38,** 2112–2120.

——, 1982: Modeling climate changes due to carbon dioxide increases. *Carbon Dioxide Review,* W. C. Clark, Ed., Oxford University Press, 104–141.

Dutton, J. A., 1995: An analytical model of atmospheric feedback and global temperature change. *J. Climate,* **8,** 1122–1139.

Fine, R. A., 1995: Tracers, time scales, and the thermohaline circulation: The lower limb in the North Atlantic Ocean. *Rev. Geophys.,* **33** (Suppl.), 1353–1365.

Hansen, J., A. Lacis, D. Rind, G. Russell, P. Stone, I. Fung, R. Ruedy, and J. Lerner, 1984: Climate sensitivity: Analysis of feedback mechanisms. *Climate Processes and Climate Sensitivity, Geophys. Monogr.,* No. 29, Maurice Ewing Vol. 5, Amer. Geophys. Union, 130–163.

——, G. Russell, A. Lacis, I. Fung, D. Rind, and P. Stone, 1985: Climate response times: Dependence on climate sensitivity and ocean mixing. *Science,* **229,** 857–859.

Harvey, L. D., and S. H. Schneider, 1985: Transient climate response to external forcing on 10^0–10^4 year time scales, Part I: Experiments with globally averaged, coupled, atmosphere and ocean energy balance models. *J. Geophys. Res.,* **90,** 2191–2205.

Hoffert, M. I., A. J. Callegari, and C.-T. Hsieh, 1980: The role of deep sea heat storage in the secular response to climatic forcing, *J. Geophys. Res.,* **85,** 6667–6679.

Houghton, J. T., G. J. Jenkins, and J. J. Ephraums, Eds., 1990: *Climate Change 1990: The IPCC Scientific Assessment.* Cambridge University Press.

——, L. G. Meira Filho, B. A. Callander, N. Harris, A. Kattenberg, and K. Maskell, Eds., 1995: *Climate Change 1995: The Science of Climate Change.* Cambridge University Press.

Kim, K.-Y., and G. R. North, 1991: Surface temperature fluctuations in a stochastic climate model. *J. Geophys. Res.,* **96,** 18 573–18 580.

——, ——, and J. Huang, 1992: On the transient response of a simple coupled climate system. *J. Geophys. Res.,* **97,** 10 069–10 081.

Kunze, E., and T. Sanford, 1996: Abyssal mixing: Where it is not. *J. Phys. Oceanogr.,* **26,** 2286–2296

Ledwell, J. R., A. J. Watson, and C. S. Law, 1993: Evidence for slow mixing across the pycnocline from an open-ocean tracer-release experiment. *Nature,* **364,** 701–703.

Munk, W., 1966: Abyssal recipes. *Deep-Sea Res.,* **13,** 707–730.

NAS, 1979: Carbon dioxide and climate: A scientific assessment. Report of an Ad Hoc Study Group on Carbon Dioxide and Climate, National Academy of Sciences, 22 pp. [Available from 2101 Constitution Ave., NW, Washington, DC 20418.]

New, A. L., R. Bleck, Y. Jia, R. Marsh, M. Huddleston, and S. Barnard, 1995: An isopycnic model study of the North Atlantic. Part I: Model experiment. *J. Phys. Oceanogr.,* **25,** 2677–2699.

Qiu, B., and R. X. Huang, 1995: Ventilation of the North Atlantic and North Pacific: Subduction versus obduction. *J. Phys. Oceanogr.,* **25,** 2374–2390.

Saravanan, R., and J. C. McWilliams, 1995: Multiple equilibria, natural variability, and climate transitions in an idealized ocean–atmosphere model. *J. Climate,* **8,** 2296–2323.

Schmitz, W. J., Jr., 1995: On the interbasin-scale thermohaline circulation. *Rev. Geophys.,* **33,** 151–173.

Schneider, S. H., and R. E. Dickinson, 1974: Climate modeling. *Rev. Geophys. Space Phys.,* **12,** 447–493.

Watts, R. G., and M. Morantine, 1990: Rapid climatic change and the deep ocean. *Climate Change,* **16,** 83–98.

Wigley, T. M. L., and M. E. Schlesinger, 1985: Analytical solution for the effect of increasing CO_2 on global mean temperature. *Nature,* **315,** 649–652.

——, and S. C. B. Raper, 1987: Thermal expansion of sea water associated with global warming. *Nature,* **330,** 127–131.